(A): Chapter 1;

(A-B): Chapter 2, Chapter 4, appendix;

(B): Chapter 3, Chapter 5.

The presentation of the material on infinite series was inspired by notes used at the Technical University of Denmark. In their original version, these notes were written by H. E. Jensen; several professors from the Department of Mathematics have contributed to the later versions. Some of our examples are borrowed from these notes.

We would like to thank a number of mathematicians who have helped us in the process of writing this book: Preben Alsholm and Ole Jørsboe for pointing out mistakes in early drafts of the manuscript; Per Christian Hansen and Torben K. Jensen for their help with the figures; Palle Jorgensen and Jakob Lemvig for useful suggestions; and Albert Cohen and Mladen Victor Wickerhauser for providing special figures. We also thank John Benedetto, ANHA series editor, for proposing that we write this book, and Hans Feichtinger for providing us with office facilities during a long-term stay at NuHAG in Vienna. Finally, we thank the staff of Birkhäuser, especially Tom Grasso, for very efficient editorial assistance; and Elizabeth Loew from TeXniques, who helped with Latex problems.

The first-named author acknowledges support from the WAVE-program, sponsored by the Danish Science Foundation.

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January 2004

In the 2nd printing, a small number of misprints are corrected. Example 3.2.6 is modified, and Exercises 2.13, 2.14, 2.16 and 3.10 are new.

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In the 3rd printing, Appendix C and Appendix D are added. Example 3.2.6 and Exercise 3.10 are modified in order to make the main idea clear and avoid technical complications. The Exercises 2.15, 2.17, 5.3, and 5.4 are new.

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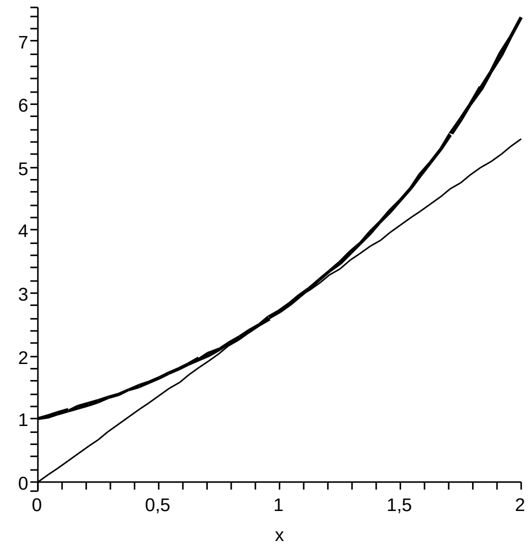

Figure 1.3.4 *The function $f(x) = e^x$ together with the first Taylor polynomial at $x_0 = 1$ (thin line).*

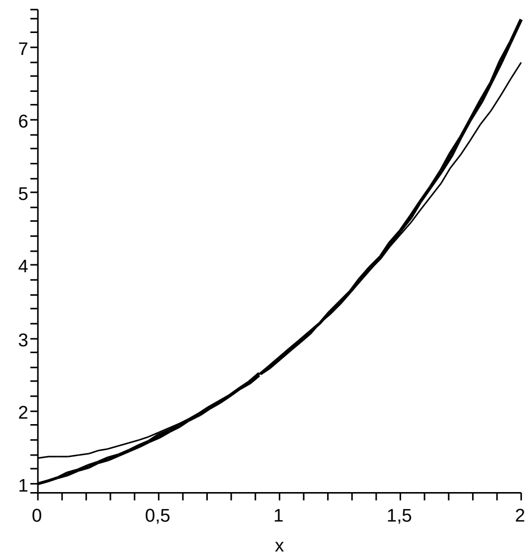

Figure 1.3.5 *The function $f(x) = e^x$ together with the second Taylor polynomial at $x_0 = 1$ (thin curve).*

This function is decreasing and continuous. If $\alpha > 1$, then

$$\int_1^t \frac{1}{x^\alpha} dx = \frac{1}{-\alpha+1}\left[x^{-\alpha+1}\right]_1^t = \frac{1}{-\alpha+1}(t^{-\alpha+1}-1).$$

It follows that

$$\int_1^t \frac{1}{x^\alpha} dx \to \frac{1}{\alpha-1} \quad \text{for } t \to \infty.$$

Via Theorem 2.2.1 we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ is convergent if } \ \alpha > 1.$$

For $\alpha = 1$,

$$\int_1^t \frac{1}{x} dx = [\ln x]_1^t = \ln t \to \infty \quad \text{for } t \to \infty.$$

Again via Theorem 2.2.1 we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}$$

Let us now find an estimate for the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$, with an error of at most $\epsilon = 0.1$. To do so, we use Corollary 2.2.2, which shows that for any $N \in \mathbb{N}$,

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{N} \frac{1}{n^2} \right| &\leq \int_{N+1}^{\infty} \frac{1}{x^2} dx + \frac{1}{(N+1)^2} \\ &= \frac{1}{N+1} + \frac{1}{(N+1)^2} \leq \frac{1}{N} \end{aligned} \tag{2.7}$$

(see Exercise 2.15). The term in (2.7) is smaller than 0.1 already for $N = 10$; this shows that with an error of maximally 0.1,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx \sum_{n=1}^{10} \frac{1}{n^2} = 1.55.$$

We shall in Example 3.6.2 prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645. \qquad \square$$

Corollary 2.2.2 only applies to series with positive terms. Another case, where an estimate for how well the finite partial sums approximate an infinite sum can be given, is for *alternating series;* by this, we mean a the sine function, is periodic with period 2π! See Figure 2.4.7, which shows the partial sum

$$S_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!};$$

this is clearly not a 2π-periodic function. This illustrates that very contra-intuitive things might happen for infinite series; we will meet several other such instances later. □

The geometric series, as well as the representations (2.15) and (2.17) derived for the exponential function and the sine function, all have the form $\sum_{n=0}^{\infty} a_n x^n$ for some coefficients a_n. In general, a series of the form $\sum_{n=0}^{\infty} a_n x^n$ is called a *power series*, and a function which can be written in the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for some coefficients a_n, is said to have a *power series representation.*

Power series appear in many contexts, where one needs to approximate complicated functions. Unfortunately not all functions can be represented with help of a power series; we will come back to this issue on page 32.

Concerning convergence of power series, a very interesting result holds: basically, it says that a power series always is convergent on a certain symmetric interval around $x = 0$ and divergent outside the interval.

Theorem 2.4.5 *For every power series $\sum_{n=0}^{\infty} a_n x^n$ one of the following options holds:*

(i) The series only converges for $x = 0$.

(ii) The series is absolutely convergent for all $x \in \mathbb{R}$.

(iii) There exists a number $\rho > 0$ such that the series is absolutely convergent for $|x| < \rho$, and divergent for $|x| > \rho$.

We give a proof of this result (under an extra assumption) in Section A.4. If the case (iii) in Theorem 2.4.5 occur, the number ρ is called the *radius of convergence* of the power series. In the case (i) we put $\rho = 0$, and in the case (ii), $\rho = \infty$. Theorem 2.4.5 also holds for complex values of x, which explains the word "radius of convergence"; see Figure 2.4.8, which corresponds to the case $\rho = 1$.

Example 2.4.6 In order to find the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we consider $x \neq 0$ and put $a_n = \frac{x^n}{n!}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \; \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 \quad \text{for } n \to \infty.$$

The quotient test shows that the series is convergent for any value of x, so $\rho = \infty$. This is in accordance with our previous observation that (2.15) holds for all x. □

Example 2.4.9 For the power series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ an argument such as in Example 2.4.6 shows that the series is convergent for $|x| < 1$ and divergent for $|x| > 1$; thus $\rho = 1$. □

In the interval given by $|x| < \rho$, it turns out that a power series defines an infinitely often differentiable function:

Theorem 2.4.10 *Assume that the power series* $\sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* $\rho > 0$, *and define the function* f *by*

$$f :]-\rho, \rho[\to \mathbb{C}, \quad f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then f *is infinitely often differentiable. Moreover*

$$f'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}, \ |x| < \rho,$$

and more generally, for every $k \in \mathbb{N}$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1)\cdots(n-k+1)x^{n-k}, \ |x| < \rho. \tag{2.19}$$

This theorem gives a part of the answer to our question about which functions have a power series expansion: in order for a function f to have such a representation, the function f must be arbitrarily often differentiable: this is a *necessary* condition. Example 2.4.12 will show that the condition is not sufficient to guarantee the desired representation. Before we present that example, we connect Theorem 2.4.10 with Taylor's theorem.

Proposition 2.4.11 *Consider a power series* $\sum_{n=0}^{\infty} a_n x^n$ *with radius of convergence* $\rho > 0$, *and let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \ x \in]-\rho, \rho[.$$

Then

$$a_n = \frac{f^{(n)}(0)}{n!}, \ n = 0, 1, \ldots, \tag{2.20}$$

i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \ x \in]-\rho, \rho[.$$

We emphasize once more that care is needed while working with infinite sums: many of the results we use for finite sums can not be generalized to infinite sums. For example, we know that any finite sum of continuous functions is continuous; the next example shows that the sum of infinitely many continuous functions does *not* need to be continuous. This demonstrates clearly that we can not just take the known rules for finite sums and use them on infinite sums.

Example 2.5.3 Consider the series

$$\sum_{n=0}^{\infty} x(1-x^2)^n.$$

We wish to find the values of $x \in \mathbb{R}$ for which the series is convergent, and to determine the sum function.

For $x = 0$ the series is convergent with sum 0. For a fixed $x \neq 0$ we can view the series as a geometric series with quotient $1 - x^2$, which is multiplied with the number x; by Theorem 2.3.3, the series is convergent for $|1 - x^2| < 1$, i.e., for $0 < |x| < \sqrt{2}$, and divergent for $|x| \geq \sqrt{2}$. For $0 < |x| < \sqrt{2}$ we obtain the sum

$$\sum_{n=0}^{\infty} x(1-x^2)^n = x\frac{1}{1-(1-x^2)} = x\frac{1}{x^2} = \frac{1}{x}.$$

Thus the sum function is

$$f(x) = \sum_{n=0}^{\infty} x(1-x^2)^n = \begin{cases} \frac{1}{x} \text{ for } 0 < |x| < \sqrt{2}, \\ 0 \text{ for } x = 0. \end{cases}$$

We see that the sum function has a discontinuity at $x = 0$ even though all functions $x(1 - x^2)^n$ are continuous! Figure 2.5.5 shows the sum function; compare with the partial sums for $N = 5$ and for $N = 50$, shown on Figure 2.5.6. □

The difference between finite and infinite sums can be used to make surprising and non-intuitive constructions. For example, one can construct a function $f : \mathbb{R} \to \mathbb{R}$, which is continuous, but nowhere differentiable:

Example 2.5.4 Given constants $A > 1, B \in]0,1[$, for which $AB \geq 1$, we attempt to consider the function

$$f(x) = \sum_{n=1}^{\infty} B^n \cos(A^n x), \; x \in \mathbb{R}.$$

Since $|B^n \cos(A^n x)| \leq B^n$ and $B \in]0,1[$, Theorem 2.1.4 combined with Theorem 2.3.3 show that the series defining f is actually convergent. A result stated formally in the next section, Theorem 2.6.5, tells us that f is continuous.

Assuming that A is an odd integer and that the product AB is sufficiently large, Weierstrass proved in 1887 that the function f is nowhere differentiable. With only classical analysis at hand, this is difficult; in Example 5.6.4 we return to this function and indicate how a short and elegant proof can be given via wavelet-inspired methods. At this moment, we only aim at an intuitive understanding of the non-differentiability. Let us consider the special case

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}. \tag{2.24}$$

The function f in (2.24) has the form (2.23) with

$$f_n(x) = \frac{\cos(2^n x)}{2^n}.$$

In a rather intuitive sense, the first terms in the series deliver the numerically largest contribution: we see that

$$|f_n(x)| \le \frac{1}{2^n} \to 0 \text{ for } n \to \infty.$$

Due to the oscillatory behavior of the functions f_n, this intuitive statement is not completely true pointwise, but if we adapt the point of view anyway, we can consider the function f as a composition of the harmonic function $f_1(x) = 1/2\cos(2x)$, see Figure 2.5.9, and the infinite sum of higher harmonics

$$\frac{1}{4}\cos(4x) + \frac{1}{8}\cos(8x) + \cdots .$$

The partial sums S_5 and S_{50} are shown in Figures 2.5.7–2.5.8. The function f_1 is shown in Figure 2.5.9, and Figure 2.5.10 shows the fifth term

$$f_5(x) = \frac{1}{32}\cos(32x);$$

already this term is very small, but it oscillates strongly. This phenomenon becomes more noticeable for larger values of n, i.e., we obtain oscillations with very high frequencies. The consequence is that the sum function f oscillates so much around each point x that it is not differentiable.

In the context of Fourier analysis we will see another indication of the relationship between differentiability and the content of oscillations; see Section 3.7. □

2.6 Uniform convergence

So far, our definition of convergence of an infinite series of functions has been *pointwise:* in fact, (2.23) merely means that if we fix an arbitrary x in

2.12 Find the values of $x \in \mathbb{R}$ for which

$$\sum_{n=0}^{\infty} \frac{1}{(2+x^2)^n}$$

is convergent, and express the sum via the standard functions.

2.13 Prove that

$$\sum_{n=k+1}^{\infty} \frac{1}{(2n-1)^2} \leq \frac{1}{4k+2} + \frac{1}{(2k+1)^2}, \ \forall k \in \mathbb{N}.$$

2.14 Find a polynomial $P(x)$ such that

$$|\sin x - P(x)| \leq 0.05 \text{ for all } \ x \in [0,3].$$

2.15 This exercise is related to Example 2.2.3, in particular to the estimate (2.7).

(i) Prove that for $N \in \mathbb{N}$,

$$\frac{1}{N+1} + \frac{1}{(N+1)^2} \leq \frac{1}{N}.$$

(ii) Find $N \in \mathbb{N}$ such that

$$\frac{4}{\pi} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq 0.01.$$

(iii) Using that $\sum_{n=N+1}^{\infty} \frac{1}{n^2} \approx \frac{1}{N+1} + \frac{1}{(N+1)^2}$, argue that for large $N \in \mathbb{N}$,

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \approx \frac{1}{N}.$$

2.16 Consider the text on the middle of page 45, "we want that the signal we transmit, i.e., $f^\sharp$, looks and behave like the original signal f". This can be formulated more exact in mathematical terms. Which kind of convergence do we need for the series $\sum_{n=0}^{\infty} a_n x^n$ – pointwise convergence for each x, or uniform convergence?

2.17 Argue that for large $J \in \mathbb{N}$,

$$\sum_{j=J}^{\infty} 2^{-j} \approx 2^{-J}.$$

3.2 Fourier's theorem and approximation

We now turn to a discussion of pointwise convergence of Fourier series. Having our experience with power series in mind, it is natural to ask for the Fourier series for a function f to converge pointwise toward $f(x)$ for each $x \in \mathbb{R}$. However, without extra knowledge about the function, this is too optimistic:

Example 3.2.1 Let $f \in L^2(-\pi, \pi)$, and define the function $g \in L^2(-\pi, \pi)$ by

$$g(x) = f(x) \text{ if } x \notin \mathbb{Z}, \; g(x) = f(x) + 1 \text{ if } x \in \mathbb{Z}.$$

Since an integral is invariant under a change of the value of the integrand in a few points, f and g have exactly the same Fourier coefficients, and therefore the same Fourier series. So at least for one of the functions, the Fourier series can not converge pointwise to the function for $x \in \mathbb{Z}$. □

Let us mention an even worse example:

Example 3.2.2 Consider the 2π-function given by

$$f(x) = \begin{cases} 1 \text{ if } x \in [-\pi, \pi[\, \cap \, \mathbb{Q}, \\ 0 \text{ if } x \in [-\pi, \pi[\, \setminus \, \mathbb{Q}. \end{cases}$$

Readers with knowledge of the Lebesgue integral can prove that all the Fourier coefficients for this function are zero. Thus the Fourier series equals zero, and does not converge to $f(x)$ if $x \in \mathbb{Q}$. □

These examples show that certain conditions are necessary if we want any pointwise relationship between a function and its Fourier series. It turns out that conditions on the smoothness of f will be sufficient in order to obtain such relationships.

A 2π-periodic function f on $\mathbb{R}$ is said to be *piecewise differentiable* if f is differentiable with a continuous derivative on the interval $]-\pi, \pi[$ — except maybe at a finite number of points $x_0, x_1, \ldots, x_n$; in a point x_j where f is non-differentiable we also require that the limits

$$\lim_{x \to x_j^+} f(x), \quad \lim_{x \to x_j^-} f(x), \quad \lim_{x \to x_j^+} f'(x), \text{ and } \lim_{x \to x_j^-} f'(x)$$

exist. For functions satisfying these conditions we have the following important result:

Proposition 3.2.5 *Assume that f is continuous, piecewise differentiable and 2π-periodic, with Fourier coefficients a_n, b_n. Then*

$$|f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|), \ \forall x \in \mathbb{R}. \tag{3.11}$$

Proof: By Theorem 3.2.3, the assumptions imply that the Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$. Via (3.7) and (3.3),

$$\begin{aligned} |f(x) - S_N(x)| &= \left| \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \right. \\ &\quad \left. - \left(\frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \right) \right| \\ &= \left| \sum_{n=N+1}^{\infty} (a_n \cos nx + b_n \sin nx) \right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n \cos nx + b_n \sin nx| \\ &\leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|). \end{aligned}$$

$\square$

In the next example we compare Theorem 3.2.3 and Proposition 3.2.5.

Example 3.2.6 Consider the 2π-periodic function given by

$$f(x) = x^2, \ x \in [-\pi, \pi[.$$

Our purpose is to find estimates for $N \in \mathbb{N}$ such that

$$|f(x) - S_N(x)| \leq 0.1 \text{ for all } x \in \mathbb{R}. \tag{3.12}$$

The reader can check (Exercise 3.10) that the Fourier series of f is given by

$$f \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2} \cos nx.$$

According to Theorem 3.2.3, the Fourier series converges uniformly to f, so we can replace the sign "$\sim$" by "$=$" .

We first apply (3.10), which was derived as a consequence of Theorem 3.2.3: it shows that (3.12) is satisfied if

$$N \geq \frac{\int_{-\pi}^{\pi}(2t)^2 dt}{\pi \cdot 0.1^2} = \frac{8\pi^3}{3\pi \cdot 0.1^2} = \frac{800\pi^2}{3} \approx 2632.$$

Let us now apply Proposition 3.2.5. Via (3.11), followed by an application

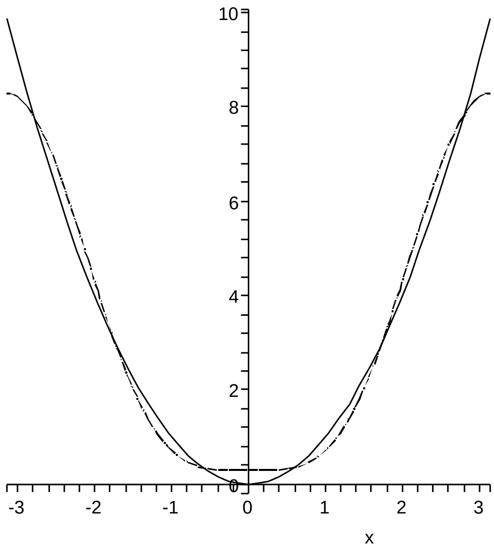

Figure 3.2.7 *The function $f(x) = x^2$ and the partial sum $S_2(x)$ (dotted), shown on the interval $[-\pi, \pi]$.*

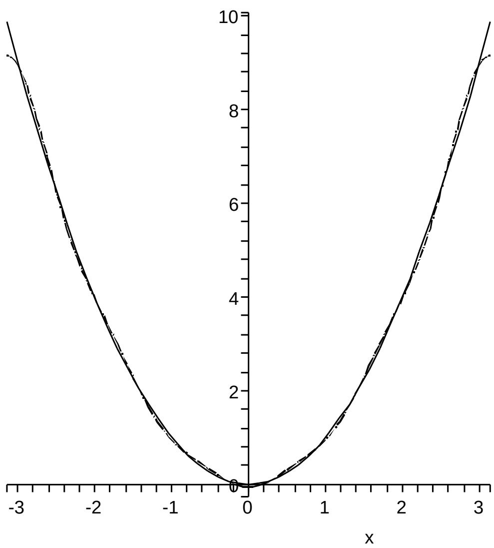

Figure 3.2.8 *The function $f(x) = x^2$ and the partial sum $S_5(x)$ (dotted), shown on the interval $[-\pi, \pi]$.*

of (2.7) in Example 2.2.3,

$$|f(x) - S_N(x)| \quad = \quad 4 \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \frac{4}{N}.$$

Thus, (3.12) is satisfied if

$$\frac{4}{N} \leq 0.1,$$

i.e., if $N \geq 40$.

For the function considered here we see that Proposition 3.2.5 leads to a much better result than Theorem 3.2.3. The difference between the estimates obtained via these two results is getting larger when we ask for better precision: if we decrease the error-tolerance by a factor of 10,

- Theorem 3.2.3 will increase the value of N by a factor of 100;
- Proposition 3.2.5 will increase the value of N by a factor of approximately 10.

See Exercise 3.10. We note that this result is based on the choice of the considered function: the difference between the use of Theorem 3.2.3 or Proposition 3.2.5 depends on the given function.

Figure 3.2.8 shows that already the partial sum S_5 gives a very good approximation to f, except for values of x which are close to $\pm\pi$. In order also to obtain good approximations close to $x = \pm\pi$, much larger values of N are needed. For example, S_{20} gives a very good approximation of f on the interval $[-3.1, 3.1]$, with a deviation not exceeding 0.005; but on the interval $[3.1, \pi]$, the error increases, and the maximal deviation is approximately 0.2. □

3.3 Fourier series and signal analysis

Fourier's theorem tells us that the Fourier series for a continuous piecewise differentiable and 2π-periodic function converges pointwise toward the function; that is, we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \ x \in \mathbb{R}. \tag{3.13}$$

Our aim is now to describe how to interpret this identity.

If we think about the variable x as time, the terms in the Fourier series correspond to oscillations with varying frequencies: for a given value of $n \in \mathbb{N}$, the functions $\cos nx$ and $\sin nx$ run through $\frac{n}{2\pi}$ periods in a time interval of unit length, i.e., they correspond to oscillations with frequency $\nu = \frac{n}{2\pi}$. Now, given f, the identity (3.13) represents f as a superposition

Theorem 3.9.1 is proved in almost all textbooks dealing with the Fourier transform, and also in several textbooks concerning signal analysis; see, e.g., [23]. Note in particular the rules (iii) and (iv); in words rather than symbols, they say that

- taking the Fourier transform of a translated version of f is done by multiplying $\hat{f}$ with a complex exponential function;
- taking the Fourier transform of a function f which is multiplied with a complex exponential function, corresponds to a translation of $\hat{f}$.

Let us show how we can use some of these rules to find the Fourier transform of a cosine function on an interval:

Example 3.9.2 Given constants $a, \omega > 0$, we want to calculate the Fourier transformation of the function (see (5.7) for the definition of $\chi_{[-\frac{a}{2},\frac{a}{2}]}$)

$$f(x) = \cos(\omega x)\chi_{[-\frac{a}{2},\frac{a}{2}]}(x). \tag{3.36}$$

This signal corresponds to an oscillation which starts at the time $x = -a/2$ and last till $x = a/2$. If x is measured in seconds, we have $\frac{\omega}{2\pi}$ oscillations per second, i.e., the frequency is $\nu = \frac{\omega}{2\pi}$. In order to find the Fourier transform, we first look at the function $\chi_{[-\frac{a}{2},\frac{a}{2}]}$ separately. Since this is an even function, Theorem 3.9.1(i) shows that for $\gamma \neq 0$,

$$\begin{aligned}
\mathcal{F}\chi_{[-\frac{a}{2},\frac{a}{2}]}(\gamma) &= 2\int_0^\infty \chi_{[-\frac{a}{2},\frac{a}{2}]}(x)\cos(2\pi x\gamma)dx \\
&= 2\int_0^{\frac{a}{2}} \cos(2\pi x\gamma)dx \\
&= \frac{2}{2\pi\gamma}\left[\sin(2\pi x\gamma)\right]_{x=0}^{x=\frac{a}{2}} \\
&= \frac{\sin \pi a\gamma}{\pi\gamma}.
\end{aligned}$$

Returning to the function f, we can use (3.25) to write

$$f(x) = \frac{1}{2}e^{i\omega x}\chi_{[-\frac{a}{2},\frac{a}{2}]}(x) + \frac{1}{2}e^{-i\omega x}\chi_{[-\frac{a}{2},\frac{a}{2}]}(x).$$

Via Theorem 3.9.1(iii) this shows that

$$\begin{aligned}
\hat{f}(\gamma) &= \frac{1}{2}\mathcal{F}\chi_{[-\frac{a}{2},\frac{a}{2}]}(\gamma - \frac{\omega}{2\pi}) + \frac{1}{2}\mathcal{F}\chi_{[-\frac{a}{2},\frac{a}{2}]}(\gamma + \frac{\omega}{2\pi}) \\
&= \frac{1}{2}\left(\frac{\sin \pi a(\gamma - \omega/2\pi)}{\pi(\gamma - \omega/2\pi)} + \frac{\sin \pi a(\gamma + \omega/2\pi)}{\pi(\gamma + \omega/2\pi)}\right).
\end{aligned}$$

Figures 3.9.3–3.9.4 show $\hat{f}$ for $\omega = 20\pi$ and different values of a in (3.36). A larger value of a corresponds to the oscillation $\cos(\omega x)$ being present in the signal over a larger time interval; we see that this increases the peak of $\hat{f}$ at the frequency $\gamma = \nu = 10$. □

3.8 (i) Prove that

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n-1)(2n+1)}, \ \ \forall x \in \mathbb{R}.$$

(ii) Calculate the number $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$.

(iii) Calculate the number $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2}$.

(iv) Write the Fourier series for $|\sin(\cdot)|$ in complex form.

(v) Compare the decay of the coefficients in the Fourier series for $\sin(\cdot)$ and $|\sin(\cdot)|$; see Theorem 3.7.2.

(vi) Denote the Nth partial sum of the Fourier series for $|\sin(\cdot)|$ by S_N. Find N such that

$$|\,|\sin x| - S_N(x)| \leq 0.1, \ \forall x \in \mathbb{R}.$$

3.9 Consider the odd 2π-periodic function, which for $x \in [0, \pi]$ is given by

$$f(x) = \frac{\pi}{96}(x^4 - 2\pi x^3 + \pi^3 x).$$

(i) Find $f(-\frac{\pi}{2})$.

(ii) Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^5}, \ \ x \in \mathbb{R}$$

(hint: $\int_0^{\pi} (x^4 - 2\pi x^3 + \pi^3 x) \sin nx \; dx = 24\frac{1-(-1)^n}{n^5}, \ n \in \mathbb{N}$).

(iii) Prove that

$$|f(x) - \sin x| \leq 0.01, \ \forall x \in \mathbb{R}$$

(hint: use the integral test).

3.10 This exercise supplements Example 3.2.6.

(i) Prove that the Fourier series in Example 3.2.6 has the announced form.

(ii) Find $N \in \mathbb{N}$ such that

$$|f(x) - S_N(x)| \leq 0.01, \ \forall x \in \mathbb{R}.$$

of numbers, functions without compact support will always have to be truncated at some place.

In practice we can not expect that ψ has all the above properties, so we need to be careful and only insist on the properties that are needed in the application we have in mind. For example, it is impossible that both ψ and its Fourier transform have compact support; this is inconvenient in cases where we need to work with the time-behavior of functions as well as their frequency-content. In such cases, we must replace the wish for compact support of either ψ or $\hat{\psi}$ with the requirement that the function at least tends very fast to zero. Formulated for the function ψ, such a requirement could be that there exist constants $C, \alpha > 0$ such that

$$|\psi(x)| \leq Ce^{-\alpha|x|}, \ \forall x \in \mathbb{R}. \tag{4.6}$$

If ψ satisfies (4.6), we say that ψ decays *exponentially.*

As already mentioned, computer-based calculations in the context of series expansions always have to be based on finite partial sums; that is, the exact representation (4.3) has to be replaced by

$$f(x) \approx \sum_{j=-N}^{N} \sum_{k=-N}^{N} c_{j,k}\psi_{j,k}(x) \tag{4.7}$$

for a sufficiently large value of $N \in \mathbb{N}$. One of the advantages of wavelets (compared with other tools leading to signal expansions) is that often it is possible to obtain a good approximation of f using only a few coefficients. Not necessarily based on a small value of N in (4.7): maybe with a relatively large value of N but with most of the coefficients $\{c_{j,k}\}_{|j|,|k|\leq N}$ vanishing. As we have seen in Example 2.7.1, it is crucial to find partial sums which approximate the given signal well, and in the context of wavelets it facilitates the use of the approximation if only a few functions $\psi_{j,k}$ appear.

The first example of a function f satisfying (4.3) was presented by Haar in his Ph.D. thesis and published in the paper [16] from 1910:

Example 4.1.7 The *Haar wavelet* is the function given by

$$\psi(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}[, \\ -1, & x \in [\frac{1}{2}, 1[, \\ 0, & x \notin [0, 1[. \end{cases} \tag{4.8}$$

The Haar wavelet satisfies the requirement that each $f \in L^2(\mathbb{R})$ has an expansion as in (4.3); the condition (4.4) is satisfied as well. Comparing with the list of desirable features on page 88, we note that ψ has compact support and is a piecewise polynomial. However, ψ is not continuous for $x = 0$, $x = \frac{1}{2}$, and $x = 1$. This might be a problem in the context of signal transmission. Recall from Section 2.7 that if we want to apply the representation (4.3) for signal transmission, the sender $\mathcal{S}$ will not be able to send the infinitely many numbers $c_{j,k}$, but only a finite set of coefficients,

5.1 Wavelets and $L^2(\mathbb{R})$

In Section 3.4 we saw that Fourier series have an exact description in terms of orthonormal bases in the Hilbert space $L^2(-\pi,\pi)$; this way of viewing Fourier series also explains the fact that the Fourier series for a given function does not necessarily converge to the function in the pointwise sense.

In wavelet analysis we are facing exactly the same situation when we want to explain the exact meaning of the representation (4.3) for functions in $L^2(\mathbb{R})$.

If we identify functions which are equal almost everywhere, we can equip $L^2(\mathbb{R})$ with the inner product

$$\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx,\ \ f,g\in L^2(\mathbb{R}), \tag{5.1}$$

and the associated norm

$$||f|| = \sqrt{\langle f,f\rangle},\ \ f\in L^2(\mathbb{R});$$

then $L^2(\mathbb{R})$ becomes a Hilbert space. The technically correct definition of a *wavelet* is that this is a function $\psi\in L^2(\mathbb{R})$ for which the functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\mathbb{R})$; translating the abstract definition in (3.16) to the setting of $L^2(\mathbb{R})$, this means that the following two sets of conditions are satisfied:

$$||f||^2 = \sum_{j,k\in\mathbb{Z}} |\langle f,\psi_{j,k}\rangle|^2,\ \forall f\in L^2(\mathbb{R}), \tag{5.2}$$

$$\langle\psi_{j,k},\psi_{j',k'}\rangle = \begin{cases} 1 \text{ if } k=k', j=j', \\ 0 \text{ otherwise.}\end{cases} \tag{5.3}$$

By (3.17) we know that this implies that each $f\in L^2(\mathbb{R})$ has the representation

$$f=\sum_{j,k\in\mathbb{Z}}\langle f,\psi_{j,k}\rangle\psi_{j,k},$$

understood in the sense that

$$\left|\left| f - \sum_{|j|,|k|\le N}\langle f,\psi_{j,k}\rangle\psi_{j,k}\right|\right| \to 0 \text{ as } N\to\infty. \tag{5.4}$$

Note that the sum in (5.4) uses a special indexing of the elements in the wavelet system. This is not crucial: one can prove that the wavelet expansion converges unconditionally in $L^2(\mathbb{R})$.

It is clear from the definition that only very special functions ψ can be wavelets. However, there exists a general framework for construction of such functions; this is the subject for the next section.

5.2 Multiresolution analysis

We already mentioned that multiresolution analysis is a central ingredient in wavelet analysis. Here is the definition:

Definition 5.2.1 *A multiresolution analysis consists of a sequence* $\{V_j\}_{j\in\mathbb{Z}}$ *of closed subspaces of* $L^2(\mathbb{R})$ *and a function* $\phi \in V_0$*, satisfying the conditions*

(i) $\cdots V_{-1} \subset V_0 \subset V_1 \cdots$*;*

(ii) $\overline{\cup_j V_j} = L^2(\mathbb{R})$ *and* $\cap_j V_j = \{0\}$*;*

(iii) $f \in V_j \Leftrightarrow [x \mapsto f(2x)] \in V_{j+1}$*;*

(iv) $f \in V_0 \Rightarrow [x \mapsto f(x-k)] \in V_0, \ \forall k \in \mathbb{Z}$*;*

(v) $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* V_0*.*

In connection with approximation theory, approximation of a function in $L^2(\mathbb{R})$ will be performed via a function from one of the spaces V_j. The importance of multiresolution analysis partly lies in the fact that the conditions guarantee the existence of convenient transforms between the spaces V_j. These transformations appear in a special case in Section 5.4.

Multiresolution analysis (abbreviated MRA) is the main subject in the majority of the published wavelet books. We will not discuss it further in this book, but only mention its relationship with the construction of wavelet bases.

The assumptions that we made in Definition 5.2.1 imply that the functions $\{2^{1/2}\psi(2x-k)\}_{k\in\mathbb{Z}}$ form an orthonormal basis for V_1. Since $\phi \in V_0 \subset V_1$, this implies that there exist coefficients $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2$ such that

$$\phi(x) = \sum_{k\in\mathbb{Z}} c_k \phi(2x-k).$$

Now, it can be proved that the function

$$\psi(x) := \sum_{k\in\mathbb{Z}} (-1)^k \overline{c_{1-k}} \phi(2x-k)$$

generates an orthonormal basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.

Without proof, we mention that the Haar wavelet can be constructed via an MRA, see Exercise 5.4.

5.3 The role of the Fourier transform

Wavelet analysis is often looked at as some kind of modern Fourier analysis. The fact that wavelet analysis has been so successful in many applications can make the reader wonder whether it eventually will replace Fourier

The wavelet transformation leads to an integral representation of functions in $L^2(\mathbb{R})$; in fact, letting

$$\psi^{a,b}(x) = \frac{1}{|a|^{1/2}}\overline{\psi(\frac{x-b}{a})}, \ a \neq 0, b \in \mathbb{R}$$

one can prove that

$$f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi(f)(a,b)\psi^{a,b}(x)\frac{dadb}{a^2}, \ f \in L^2(\mathbb{R})$$

(we shall not go into details with the exact meaning of the integral). The admissibility condition needed to make this work is most conveniently expressed via the Fourier transform, in form of the condition

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} d\gamma < \infty. \qquad \square$$

5.4 The Haar wavelet

We already mentioned that the first wavelet ever constructed was the Haar wavelet,

$$\psi(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}[, \\ -1, & x \in [\frac{1}{2}, 1[, \\ 0, & x \notin [0, 1[. \end{cases} \tag{5.6}$$

In the entire section, ψ will denote this particular wavelet. The function ψ and the scaled and translated versions $\psi_{j,k}$ are called *Haar functions*.

Given any interval $I \subset \mathbb{R}$, the associated *characteristic function* χ_I is defined by

$$\chi_I(x) = \begin{cases} 1, & x \in I, \\ 0, & x \notin I. \end{cases} \tag{5.7}$$

In terms of characteristic functions, the Haar wavelet can be written

$$\psi(x) = \chi_{[0,\frac{1}{2}[}(x) - \chi_{[\frac{1}{2},1[}(x).$$

In the entire section we consider a continuous function

$$f : [0, 1] \to \mathbb{R}.$$

Our aim is to approximate such a function via piecewise constant functions. As we will see, the Haar wavelet shows up in our attempt to do so. Our final goal is to prove Theorem 5.4.13 and Theorem 5.4.14.

A natural way to approximate f is to split $[0, 1]$ into a certain number of intervals, and then, on each of these intervals, approximate $f(x)$ by the average of f over the interval. We will focus on the case where we split $[0,1]$ into 2^k intervals for some $k = 0, 1, \ldots$; we will further choose the intervals to have equal length. Ignoring the endpoint $x = 1$, this means that our intervals are

$$I_n = [n2^{-k}, (n+1)2^{-k}[, \; n = 0, 1, \ldots, 2^k - 1.$$

These intervals have length 2^{-k}, so the average of f on I_n is

$$a_n = 2^k \int_{n2^{-k}}^{(n+1)2^{-k}} f(x)dx. \tag{5.8}$$

Note that the intervals I_n as well as the coefficients a_n actually depend on the chosen value of k. Thus, it would have been more accurate to denote the intervals by $I_{k,n}$ and the coefficients by $a_{k,n}$; however, except in Theorem 5.4.13 we will suppress the dependence on k.

If k is sufficiently large (i.e., if the intervals I_n are sufficiently small), the value a_n is a reasonable approximation to $f(x)$ for $x \in I_n$; thus, as an approximation of f on $[0,1]$ it is natural to take

$$f_k(x) := \sum_{n=0}^{2^k-1} a_n \chi_{I_n}(x) = \sum_{n=0}^{2^k-1} a_n \chi_{[n2^{-k},(n+1)2^{-k}[}(x). \tag{5.9}$$

The parameter k indicates the *level,* or *scale,* of the approximation: large values of k lead to fine approximations, while small values give us coarse approximations which are constant on large intervals. By choosing k sufficiently large, we can obtain that f_k approximates f as well as we would like in the uniform sense:

Lemma 5.4.1 *Assume that $f : [0,1] \to \mathbb{R}$ is continuous. Then, for any $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that*

$$|f(x) - f_k(x)| \le \epsilon, \; \forall x \in [0,1[,$$

for all $k \ge K$.

Proof: A continuous function on a bounded and closed interval is uniformly continuous, i.e., for any given $\epsilon > 0$ we can choose $\delta > 0$ such that

$$|x - y| \le \delta, \; x, y \in [0,1] \Rightarrow |f(x) - f(y)| \le \epsilon.$$

If we now choose $K \in \mathbb{N}$ such that $2^{-K} \le \delta$, then $2^{-k} \le \delta$ for all $k \ge K$. Thus, for such k, the maximal variation of $f(x)$ on each interval I_n is at most ϵ. Now, considering an arbitrary $x \in [0,1[$, this point belongs to exactly one of the intervals, say, I_n, and therefore

$$|f(x) - f_k(x)| = |f(x) - a_n|.$$

Since a_n is an average of function values which deviate at most ϵ from $f(x)$, we obtain the announced result. $\square$

Fourier series where n ranges from $-N$ to N for some $N \in \mathbb{N}$, then

$$\begin{aligned}
\left\| f - \sum_{|n|\le N} c_n e^{inx} \right\|^2 &= \left\| \sum_{|n|>N} c_n e^{inx} \right\|^2 \\
&= \sum_{|n|>N} |c_n|^2 \\
&\sim \sum_{n=N+1}^{\infty} \frac{1}{n^2} \\
&\sim \frac{1}{N} \qquad \text{(See Exercise 2.15).}
\end{aligned}$$

A reformulation of this result says that approximation with N Fourier coefficients leads to a square-error of the size $1/N$ (times a constant).

Let us now consider approximation via the Haar wavelet. One can prove that the following argument, based on (5.24), is correct, despite the fact that f is discontinuous. Looking at the representation of f in (5.24), the natural thing to do is to order the infinite sum by starting with the small values of j. If we only have capacity to calculate or store $N = 2^J$ coefficients for some $J \in \mathbb{N}$, this means that we can consider $j = 0, 1, \dots, J-1$; this will use

$$\sum_{j=0}^{J-1} \sum_{n=0}^{2^j-1} 1 = 1 + 2 + \cdots + 2^{J-1} = 2^J - 1 = N - 1$$

coefficients. For each of these values of j, only one of the coefficients in $\sum_{n=0}^{2^j-1} \langle f, \psi_{j,n}\rangle \psi_{j,n}(x)$ is nonzero, namely, the one corresponding to the value of n for which $x_0 \in I_{j,n}$. According to Example 5.6.1, this particular coefficient has the size $\langle f, \psi_{j,n}\rangle \sim 2^{-j/2}$; thus, approximating with N coefficients leads to an error of the size

$$\begin{aligned}
\left\| f - \langle f, \chi_{[0,1[}\rangle \chi_{[0,1[} - \sum_{j=0}^{J-1} \sum_{n=0}^{2^j-1} \langle f, \psi_{j,n}\rangle \psi_{j,n} \right\|^2 &= \left\| \sum_{j=J}^{\infty} \sum_{n=0}^{2^j-1} \langle f, \psi_{j,n}\rangle \psi_{j,n} \right\|^2 \\
&= \sum_{j=J}^{\infty} \sum_{n=0}^{2^j-1} |\langle f, \psi_{j,n}\rangle|^2 \\
&\sim \sum_{j=J}^{\infty} 2^{-j} \\
&\sim \frac{1}{2^J} \text{ (See Exercise 2.17)} \\
&= \frac{1}{N}.
\end{aligned}$$

The conclusion is that wavelet approximation leads to an approximation error of the same size as Fourier analysis. The picture changes completely if we consider nonlinear approximation via wavelets. As already noticed, for each level j, exactly one coefficient is nonzero: if we have capacity to calculate $N = 2^J$ coefficients, we can simply calculate the nonzero coefficients associated to the first N levels. This leads to an approximation error of the size

$$\begin{aligned}
&\left\|f - \langle f, \chi_{[0,1[}\rangle \chi_{[0,1[} - \sum_{j=0}^{N}\sum_{n=0}^{2^j-1} \langle f, \psi_{j,n}\rangle \psi_{j,n}\right\|^2 \\
= &\left\| \sum_{j=N+1}^{\infty}\sum_{n=0}^{2^j-1} \langle f, \psi_{j,n}\rangle \psi_{j,n}\right\|^2 \\
= &\sum_{j=N+1}^{\infty}\sum_{n=0}^{2^j-1} |\langle f, \psi_{j,n}\rangle|^2 \\
\sim &\sum_{j=N+1}^{\infty} 2^{-j} \\
\sim &\; 2^{-N-1}.
\end{aligned}$$

That is, nonlinear approximation leads to exponential decay of the error, which is a vast improvement compared to the standard wavelet method. □

5.8 Frames

The purpose of this section is to discuss a generalization of the wavelet theory presented so far; let us first explain why we want to do so.

So far, we have been discussing wavelets ψ satisfying the two conditions (4.3) and (4.4), or, (5.2) and (5.3). However, just a glance at these conditions reveals that they can only be satisfied for very special choices of ψ. For example, (5.3) means that

$$\int_{-\infty}^{\infty} 2^{j/2+j'/2}\psi(2^j x - k)\overline{\psi(2^{j'} x - k')}dx = \begin{cases} 1 \text{ if } j = j', k = k', \\ 0 \text{ otherwise.} \end{cases}$$

It certainly needs some work to find ψ such that just this condition is satisfied. While wavelet analysis actually tells us how we can satisfy the two sets of conditions, it is also clear that it might be difficult (or even impossible) to construct wavelets having additional desirable features. A case where such a limitation appears was pointed out already by Daubechies in her book [8]:

Theorem 5.9.3 *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. If the Gabor system* (5.38) *is an orthonormal basis for $L^2(\mathbb{R})$, then*

$$\left(\int_{-\infty}^{\infty} |xg(x)|^2 dx\right)\left(\int_{-\infty}^{\infty} |\gamma \hat{g}(\gamma)|^2 d\gamma\right) = \infty. \tag{5.40}$$

For a proof of the Balian–Low theorem we refer to [18]. In words, the Balian–Low theorem means that a function g generating a Gabor basis can not be well localized in both time and frequency, in the sense that both g and $\hat{g}$ have compact support or decay quickly. For example, it is not possible that g and $\hat{g}$ satisfy estimates like

$$|g(x)| \leq \frac{C}{1+x^2}, \quad |\hat{g}(\gamma)| \leq \frac{C}{1+\gamma^2}$$

simultaneously.

If fast decay of g and $\hat{g}$ is needed, we have to ask whether we need all the properties characterizing an orthonormal basis or whether we can relax some of them. The property we want to keep is that every $f \in L^2(\mathbb{R})$ has an expansion in terms of modulated and translated versions of the function g, as in (5.39). It turns out that the expansion property actually can be combined with g and $\hat{g}$ having very fast decay: what we have to do is to allow the Gabor system to be a frame rather than an orthonormal basis:

Example 5.9.4 Let $a, b > 0$ and consider the *Gaussian*, i.e., the function $g(x) = e^{-x^2}$. Then one can prove (it is difficult and not meant to be an exercise!) that the generated Gabor system is a frame if and only if $ab < 1$. The frames we obtain for $ab < 1$ are very well localized in time (by their definition) and in frequency, because

$$\hat{g}(\gamma) = \sqrt{\pi} e^{-\pi^2\gamma^2}. \qquad \square$$

As we have seen several times before, practical calculations always have to be performed on finite systems of functions; for Gabor systems this means that at a certain point we will have to restrict our attention to a finite family of the form

$$\left\{e^{2\pi imbx} g(x - na)\right\}_{|m|,|n| \leq N} \tag{5.41}$$

(or a subset hereof) for some $N \in \mathbb{N}$. This family spans a finite-dimensional subspace V of $L^2(\mathbb{R})$, and we can deal with it via linear algebra. An interesting and somehow surprising fact is that the vectors always form a basis for V, i.e., they are automatically linearly independent! This result was proved in some special cases by Heil et al. in [17] and conjectured to hold generally; the full proof was given later by Linnell [24].

However, as a final comment we note that the original conjecture by Heil et al. is even more general: it claims that if $\{(\mu_k, \lambda_k)\}_{k=1}^N$ is an arbitrary collection of distinct points in $\mathbb{R}^2$ and $g \neq 0$, then the *generalized Gabor system*

$$\left\{e^{2\pi i \lambda_k x} g(x-\mu_k)\right\}_{k=1}^{N}$$

is linearly independent. This conjecture is still open, despite the fact that several researchers have tried hard to prove or disprove it.

5.10 Exercises

5.1 Consider the function

$$f(x) = \sin \pi x \ \chi_{[0,1]}(x).$$

(i) Make drafts (e.g., via Maple or Mathematica) of some of the functions f_k introduced in Section 5.4 and compare with f.

(ii) Compare the approximations in (i) with the partial sums of the power series for the function $x \mapsto \sin \pi x$.

5.2 Consider the function

$$f(x) = e^x \ \chi_{[0,1]}(x).$$

(i) Make drafts (e.g., via Maple or Mathematica) of some of the functions f_k introduced in Section 5.4 and compare with f.

(ii) Compare the approximations in (i) with the partial sums of the power series for the exponential function.

(iii) Compare the approximations in (i) with the partial sums of the Fourier series for the 2π-periodic function given by

$$g(x) = e^x, \ x \in [-\pi, \pi[.$$

5.3 Use the definition on page 106 to prove that the Gaussian

$$\psi(x) = e^{-x^2}$$

is not a wavelet.

5.4 Let $\phi(x) = \chi_{[0,1]}$ and put $V_0 = \overline{\text{span}}\{f(\cdot - k) : \ k \in \mathbb{Z}\}$.

(i) Describe the vector spaces V_j formed by condition (iii) in Definition 5.2.1.
(ii) Show that the conditions in Definition 5.2.1 are satisfied.

Remark: The multiresolution analysis based on $\phi(x) = \chi_{[0,1]}$ leads to the Haar wavelet, see Section 5.4.

Appendix C

In this appendix we collect the formulas for Fourier series for functions with period T for some $T > 0$.

C.1 Fourier series for T-periodic functions

Given $T > 0$, let $\omega = 2\pi/T$. For a function $f : \mathbb{R} \to \mathbb{C}$ with period T, which belongs to the vector space

$$L^2(-\frac{T}{2}, \frac{T}{2}) := \left\{ f : \quad \int_{-T/2}^{T/2} |f(x)|^2 dx < \infty \right\},$$

the Fourier series is

$$f \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x),$$

where

$$a_n = \frac{2}{T} \int_0^T f(x) \cos n\omega x \; dx, \;\; n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{2}{T} \int_0^T f(x) \sin n\omega x \; dx, \;\; n = 1, 2, \ldots \;\; .$$

The Nth partial sum of the Fourier series is

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos n\omega x + b_n \sin n\omega x).$$

Similar to Theorem 3.1.1, we have that

(i) If f is an even function, then $b_n = 0$ for all n, and

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos n\omega x \; dx, \;\; n = 0, 1, 2, \dots \;\; .$$

(ii) If f is odd, then $a_n = 0$ for all n, and

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\sin n\omega x \; dx, \;\; n = 1, 2, \dots \;\; .$$

The Fourier series of f in complex form is

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega x},$$

where

$$c_n = \frac{1}{T}\int_0^{T} f(x)e^{-in\omega x}dx, \quad n \in \mathbb{Z}.$$

Under assumptions similar to the one used in Theorem 3.2.3, the maximal deviation between $f(x)$ and the partial sum $S_N(x)$ can be estimated by

$$|f(x) - S_N(x)| \le \frac{1}{\pi}\sqrt{\frac{T}{2N}\int_0^{T} |f'(t)|^2 dt}.$$

Alternatively, as in Proposition 3.2.5,

$$|f(x) - S_N(x)| \le \sum_{n=N+1}^{\infty} (|a_n| + |b_n|).$$

Parseval's Theorem takes the form

$$\frac{1}{T}\int_0^{T} |f(x)|^2 dx = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Appendix D

Infinite series appear in many fields of science. For example, modelling of an electric circuit or a mechanical system often leads to a differential equation, which has a solution given in terms of an infinite series. In this appendix we give a short description of the so-called *Fourier's method,* which can be used to find a solution to a differential equation with a periodic input.

D.1 Fourier's method

Consider a linear nth order differential equations of the type

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = u(x), \; x \in I. \tag{D.1}$$

Here $a_0, \ldots, a_n \in \mathbb{R}$ are given constants; $I \subset \mathbb{R}$ is an open interval, and $u : I \to \mathbb{R}$ is a given continuous function, called the *input.* We search for a function y, which solves the differential equation; such a function is called an *output* or a *response.*

An equation of the type (D.1) has infinitely many solutions. On the other hand, for any set of *initial conditions*, i.e., prescribed values of the numbers

$$y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0),$$

there is a unique solution.

If y is any particular solution to (D.1) and y_h is the general solution to the corresponding homogeneous differential equation, then $y + y_h$ is the general solution to (D.1). It is known how to solve the homogeneous equation, so the problem is how to find a particular solution. Our purpose is to show that if u is 2π-periodic, this can be done via Fourier series. The idea is simple: due to the periodicity assumption, we can expand u in terms of the complex exponential functions $e^{inx}, n \in \mathbb{Z}$, i.e., in a complex Fourier series. By a direct calculation we can solve the differential equation (D.1) for each of the inputs e^{inx}; by superposition, this leads to a solution for the given input u.

The *characteristic polynomium* for (D.1) is defined by

$$P(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n.$$

By direct verification, the reader can prove the following.

Theorem D.1.1 *Let $s \in \mathbb{C}$, and assume that $P(s) \neq 0$. If $u(x) = e^{sx}$, then* (D.1) *has exactly one solution of the type*

$$y(x) = H(s)e^{sx}; \tag{D.2}$$

this solution is obtained for

$$H(s) = \frac{1}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}. \tag{D.3}$$

The function H defined by (D.3) is called the *transfer function* associated with (D.1). By definition, it can be used to solve the differential equation (D.1) for special inputs: if

$$u(x) = e^{inx}$$

for some $n \in \mathbb{Z}$, where $P(in) \neq 0$, then (D.1) has the solution

$$y(x) = H(in)e^{inx}.$$

Now consider a function u which is 2π-periodic, piecewise differentiable, and continuous. By Theorem 3.2.3, u equals its Fourier series; writing the Fourier series in complex form, we have

$$u(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad x \in \mathbb{R}, \tag{D.4}$$

The Nth partial sum of the Fourier series is

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}.$$

If $P(s) \neq 0$ for all $s = in, n = -N, \ldots, N$, then we can find a solution to (D.1) with input S_N, namely

$$y(x) = \sum_{n=-N}^{N} c_n H(in) e^{inx}.$$

Now it is natural to ask whether the input u in (D.4) leads to the solution

$$y(x) \sim \sum_{n=-\infty}^{\infty} c_n H(in) e^{inx}, \quad x \in \mathbb{R}. \tag{D.5}$$

It turns out that the answer is affirmative if a technical condition is satisfied. We need the following definition.

Definition D.1.2 *The differential equation* (D.1) *is asymptotically stable if all the solutions to the equation $P(s) = 0$ have a negative real part.*

Theorem D.1.3 *Assume that* (D.1) *is asymptotically stable, and let H denote the transfer function. Let u be a 2π-periodic piecewise differentiable and continuous function, given by the Fourier series*

$$u(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad x \in \mathbb{R}.$$

Then (D.1) *has a solution, given by the Fourier series*

$$y(x) = \sum_{n=-\infty}^{\infty} H(in) c_n e^{inx}, \quad x \in \mathbb{R}.$$

We note that the obtained solution is given via a Fourier series, i.e., it is not given explicitly in terms of classical functions. Very often it is impossible to give an explicit expression for the solution y, i.e., we have to work with an "unknown function" which is only "known" via its Fourier series. For such a case, it is very important that we have developed methods to determine how many terms we need to use in the Fourier series in order to obtain a certain precision, see Theorem 3.2.3 and Proposition 3.2.5.

Example D.1.4 Consider the differential equation

$$\frac{dy}{dt} + y = u, \tag{D.6}$$

where u is the 2π-periodic function given by

$$u(x) = |x|, \ x \in [-\pi, \pi[.$$

We note that the characteristic polynomial for (D.6) is

$$P(s) = s + 1;$$

since $P(s) = 0 \Leftrightarrow s = -1$, the differential equation is asymptotically stable. The transfer function is

$$H(s) = \frac{1}{1+s}, \ s \neq -1.$$

The reader can check that the Fourier series of the function u in complex form is

$$u(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where

$$c_n = \begin{cases} \frac{\pi}{2} & \text{if } n = 0, \\ 0 & \text{if n is even, } n \neq 0, \\ -\frac{2}{\pi}\frac{1}{n^2} & \text{if n odd.} \end{cases}$$

According to Theorem D.1.3 we obtain a solution to (D.6), whose Fourier series is given by

$$y(x) = \sum_{n=-\infty}^{\infty} \frac{c_n}{1+in} e^{inx}, \tag{D.7}$$

The function y in (D.7) is not given explicitly, so we need our methods from Fourier analysis to find out how many terms we need in order to obtain a reasonable approximation. Via calculations exactly like in the proof of Proposition 3.2.5 we see that

$$\begin{aligned} \left| y(x) - \sum_{n=-N}^{N} \frac{c_n}{1+in} e^{inx} \right| &= \left| \sum_{|n|>N} \frac{c_n}{1+in} e^{inx} \right| \\ &\leq \sum_{|n|>N} \left| \frac{c_n}{1+in} \right| \\ &\leq \sum_{|n|>N} \frac{|c_n|}{\sqrt{1+n^2}} \\ &\leq \frac{4}{\pi} \sum_{n=N+1}^{\infty} \frac{1}{n^2} && \text{(D.8)} \\ &\leq \frac{4}{\pi}\frac{1}{N} && \text{(D.9)} \end{aligned}$$

(see (2.7)). For $N \geq 128$, the expression in (D.9) is smaller than 0.01; we conclude that for $N \geq 128$,

$$\left| y(x) - \sum_{n=-N}^{N} \frac{c_n}{1+in} e^{inx} \right| \leq 0.01, \ \forall x \in \mathbb{R}. \qquad \square$$

List of Symbols

$\mathbb{R}$:	The real numbers.
$\mathbb{N}$:	The natural numbers: 1,2,3,....
$\mathbb{Z}$:	The integers.
$\mathbb{Q}$:	The rational numbers.
$\mathbb{C}$:	The complex numbers.
i :	The complex unit number.
$\lvert x\rvert$:	Absolute value of complex number x.
$\overline{x}$:	The complex conjugate of $x \in \mathbb{C}$.
$\mathcal{H}$:	Hilbert space.
$L^p(\mathbb{R})$:	For $p \in [1, \infty[$, the space of measurable functions $f : \mathbb{R} \mapsto \mathbb{C}$ for which $\int_{\mathbb{R}} \lvert f(x)\rvert^p dx < \infty$.
$f^{(k)}(x)$:	The kth derivative of the function f.
$C^k(\mathbb{R})$:	The space of k times differentiable functions with a continuous kth derivative.
$\mathcal{F}f = \hat{f}$:	The Fourier transform, for $f \in L^1(\mathbb{R})$ given by $\hat{f}(\gamma) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\gamma}dx$.
χ_A :	The characteristic function for a set A, $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$.
$\overline{A}$:	The closure of a set A.
$\text{supp}f$:	The support of the function f: $\text{supp}f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$.
T_a :	The translation operator $(T_a f)(x) = f(x - a)$.
$\langle \cdot, \cdot \rangle$:	The inner product in a Hilbert space.
$\lvert\lvert \cdot \rvert\rvert$:	The norm in a normed vector space.

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Index

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Approximation Theory
From Taylor Polynomials to Wavelets
Christensen, O.; Christensen, K.L.
2005, XI, 156 p. 5 illus., Softcover
ISBN: 978-0-8176-3600-5
A product of Birkhäuser Basel